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Engineering Notes

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Algebraic Riccati-Equation-Based Differentiation Trackers

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Introduction

OFTEN we would like to estimate the state of a system given a set of measurements taken over an interval of time. Kalman filtering is a useful technique for estimating or updating the previous estimate of a system's state by 1) using indirect measurements of the state variables and 2) using the covariance information of both the state variables and the indirect measurements. Kalman filtering[1] originally started as a solution to the state estimation problem in linear time-invariant state-space structures, in which a linear, stochastic dynamic system is represented by a set of differential equations describing the evolution of the states in time and a set of algebraic equations that map the states to measured outputs. Today Kalman filtering is one of the key tools in data assimilation, and it has been used extensively in many diverse applications, such as navigation and guidance systems, radar tracking, sonar ranging, and satellite orbit determination.[2]

The Kalman filter requires knowledge of the process dynamics, and the quality of the filter estimation depends on the degree of accuracy of the system model. The goal of this Note is to present a continuous-time Kalman-filter-based differentiator that estimates the higher derivatives of a given output without knowledge of the system dynamics being observed. A discrete-time version will be deduced from the continuous-time observer. This technique can be, of course, generalized to any dynamic system. Consequently, under certain mild assumptions, the proposed differentiators can be used as key elements to reproduce any unknown state that is given as a function of the input, the output, and finite number of their higher derivatives.[3] To highlight the effectiveness of such an observation approach, note that behavior of a state variable of a dynamic system cannot be determined exactly by direct measurements; instead, we usually find that the measurements that we make are functions of the state variables and that these measurements are corrupted by random noise. The system itself may also be subjected to random disturbances. It is then required to estimate the state variables from noisy observations. See Refs. 3–11 for past work in this area.

Notice that a lot of nonlinear observation techniques described in the literature cannot operate directly on the data without any knowledge about the system model being observed. To clarify this, consider the two-dimensional nonlinear system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = f(x_1, x_2, \theta), \qquad y = x_1 \tag{1}$$

where $f(\cdot)$ is a nonlinear function of the state variables and the unknown parameter θ.

Constructing a classical nonlinear observer for Eq. (1) necessitates at least the knowledge of the upper bound of $f(\cdot)$ but if we are able to observe or estimate $x_2 = \dot{y}$ from only the knowledge of y, this implies that the differentiation procedure is insensitive to the presence of the uncertain parameter θ. This is not to say that differentiation observer algorithms always exclude knowledge of the system dynamics, but, rather, that they can operate with or without this knowledge (depending on the structure of the system).[3,7–9] Depending on the observation constraints, the user decides what types of observation techniques will be used and when they will be used. See Ref. 3, where we have examined several examples showing that the differentiation observer either uses, in a fundamentally essential way, the dynamics of the system or only uses observational data information.

The question is why have we chosen the structure of the Kalman filter to build an arbitrary-order differentiator. This is for the following reasons: 1) The arbitrary-order differentiator can be used without any proof of convergence or synthesis of its stability. 2) The differentiator representation in a state-space form offers a remarkable simplicity while we use the differentiator in closed-loop configuration. 3) The Kalman-filter-based differentiator takes into account the measurement uncertainties explicitly. 4) The Kalman-filter-based differentiator takes measurements into account incrementally. 5) The Kalman filter based differentiator can take into account a priori information, if any.

One of the interesting applications of numerical differentiation is to track the target motion with an unknown dynamic model.[12–15] Tracking a target of unknown varying model is a difficult task. A common technique to accomplish this task is a Kalman filter, which predicts the state of the target at the next time period and chooses the detection that best matches this predicted state. The fundamental problems in target tracking are 1) the absence of accurate target models and b) noise in the measurements. If the target dynamic model were known, it would be then possible to build an observer, and the problem would be completely solved. However, this is not possible, and hence, accurate tracking necessitates robust time-derivative estimation of the measured angles, such as the azimuth, etc. Consequently, the tracking problem will be reduced to a classical differentiation problem. Throughout this Note $\mathbb{R}$ is the set of real numbers, $\|\cdot\|$ denotes the habitual Euclidean norm, $\lambda_{\max}(A)$ is the largest eigenvalue of the matrix A, $\lambda_{\min}(A)$ is the smallest eigenvalue of A, and $\delta_{i,j}$ is the Kronecker symbol.

Continuous-Time Differentiation Observer

Refer to the vast literature that has been devoted to target tracking with Kalman filter; differentiation algorithms have been developed in discrete-time models. These models are designed to estimate the velocity and the acceleration of the target in certain coordinates and are known as the α–β tracker and the α–β–γ tracker. In this Note, we start by developing the nth-order continuous-time differentiation observer, then under certain conditions, we extend the theory to the discrete-time case.

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This section is mainly concerned with the derivation of the Kalman-filter-based differentiation algorithm from the point of view of it being a time-varying linear differentiator and with how the differentiation algorithm may be used in practice.

Because there is no model for the measured output y, which is supposed to be of finite energy with its higher derivatives, we will estimate the $(n-1)$ first derivatives of y using a time-varying linear system. This system has the form of a classical Kalman filter. The whole design of this differentiator is given by the following theorem.

Theorem 1. Consider the continuous-time system

$$\dot{x} = Ax + P(t)C'(y - Cx)$$
$$\dot{P}(t) = AP(t) + P(t)A' - P(t)C'CP(t) + Q(\alpha, \gamma) \tag{2}$$

where

$$A \in \mathbb{R}^{n\times n}, \qquad A_{i,j} = \delta_{i,j-1}, \qquad 1 \le i, j \le n$$
$$C \in \mathbb{R}^{1\times n}, \qquad C_i = \delta_{1,i}$$
$$Q(\alpha,\gamma) = \mathrm{diag}\Big[\big(\alpha_i^2 - 2\alpha_{i+1}\alpha_{i-1} + 2\alpha_{i+2}\alpha_{i-2} - \cdots\big)\gamma^{2i} > 0, \quad i = 1,\ldots,n\Big] \in \mathbb{R}^{n\times n} \tag{3}$$

with $\alpha_0 = 1$ and $\alpha_i = 0$ for $0 > i > n$. If 1) y is of class $\mathscr{C}^{(n)}$ and $\sup_{t\ge 0}|y^{(i)}| < \gamma$, $1 \le i \le n$, and 2) the polynomial

$$s^n + \sum_{i=1}^{n}\alpha_i s^{n-i}$$

is Hurwitz, then for large values of γ system (2) approximates the successive higher derivative of the signal y when time elapses.

Proof. Let $\boldsymbol{x}_\infty$ be the vector state of the following system:

$$\dot{\boldsymbol{x}}_\infty = A\boldsymbol{x}_\infty + P_\infty C'(y - C\boldsymbol{x}_\infty)$$
$$AP_\infty + P_\infty A' - P_\infty C'CP_\infty + Q(\alpha,\gamma) = 0 \tag{4}$$

and let $e = x - \boldsymbol{x}_\infty$, then

$$\dot{e} = Ae + P(t)C'(y - Cx) - P_\infty C'(y - C\boldsymbol{x}_\infty)$$

By adding and subtracting the term $P(t)C'C\boldsymbol{x}_\infty$ to the last equation, we have

$$\dot{e} = [A - P(t)C'C]e + [P(t) - P_\infty]C'(y - C\boldsymbol{x}_\infty) \tag{5}$$

Note that $w = y - C\boldsymbol{x}_\infty$ and $b = [P(t) - P_\infty]C'$. Because $\mathrm{d}/\mathrm{d}t[P(t)P^{-1}(t)] = 0$, then $\dot{P}^{-1}(t) = -P^{-1}(t)\dot{P}(t)P^{-1}(t)$. Using Eqs. (2), we have

$$\dot{P}^{-1}(t) = -P^{-1}(t)A - A'P^{-1}(t) - P^{-1}(t)Q(\alpha,\gamma)P^{-1} + C'C \tag{6}$$

Then if we take $V = e'P^{-1}(t)e$ as a Lyapunov function for Eq. (5), we get

$$\begin{aligned}\dot{V} &= \dot{e}'P^{-1}(t)e + e'\dot{P}^{-1}(t)e + e'P^{-1}(t)\dot{e} = \{e'[A' - C'CP(t)] \\ &\quad + w'b'\}P^{-1}(t)e + e'[-P^{-1}(t)A - A'P^{-1}(t) \\ &\quad - P^{-1}(t)Q(\alpha,\gamma)P^{-1}(t) + C'C]e \\ &\quad + e'P^{-1}(t)\{[A - P(t)C'C]e + bw\}\end{aligned}$$

This gives

$$\begin{aligned}\dot{V} &= e'[-P^{-1}(t)Q(\alpha,\gamma)P^{-1}(t) - C'C)e + 2e'P^{-1}(t)bw \\ &\le e'[-P^{-1}(t)Q(\alpha,\gamma)P^{-1}(t)]e + 2e'P^{-1}(t)bw\end{aligned}$$

Let $\xi = P^{-1}(t)e$, then $V = \xi'P(t)\xi$ and

$$\begin{aligned}\dot{V} &\le -\xi'Q(\alpha,\gamma)\xi + 2\xi'bw \\ &\le -\lambda_{\min}[Q(\alpha,\gamma)]\|\xi\|^2 + 2\|\xi\|\|b\|\|w\|\end{aligned}$$

Because $P(t)$ is strictly increasing and $\lim_{t\to\infty}P(t) = P_\infty$, then there exists $\mu > 0$ such that

$$\|b\| \le \|P(0) - P_\infty\|e^{-\mu t}$$

Furthermore, for large values of γ and $\boldsymbol{x}_\infty(0) \ll \gamma$, the error function $y - C\boldsymbol{x}_\infty$ is bounded as follows:

$$\sup_{t\ge 0}|y - C\boldsymbol{x}_\infty| = \sup_{t\ge 0}\bigg| y - C\exp[(A - P_\infty C'C)t]\boldsymbol{x}_\infty(0) - C\int_0^t \exp[(A - P_\infty C'C)(t-\tau)]P_\infty C'y(\tau)\,\mathrm{d}\tau\bigg| \le \sup_{t\ge 0}|y| \le \gamma$$

We have $(\alpha_1^2 - 2\alpha_2)\gamma^2\|\xi\|^2 \le \xi'Q(\alpha,\gamma)\xi \le \alpha_n\gamma^{2n}\|\xi\|^2$, then

$$\begin{aligned}\dot{V} &\le -\lambda_{\min}[Q(\alpha,\gamma)]\|\xi\|^2 + 2c_1\gamma\|\xi\|e^{-\mu t} \\ &\le -\frac{\lambda_{\min}[Q(\alpha,\gamma)]}{2}\|\xi\|^2 + \frac{2c_1^2\gamma^2}{\lambda_{\min}[Q(\alpha,\gamma)]}e^{-2\mu t} \\ &\le -C_1\|\xi\|^2 + C_2 e^{-2\mu t}\end{aligned}$$

where $c_1 = \|P(0) - P_\infty\|$, $C_1 = (\alpha_1^2 - 2\alpha_2)\gamma^2/2$, and $C_2 = 2c_1^2/(\alpha_1^2 - 2\alpha_2)$. From the last inequality, we have

$$\dot{V} \le -C_3 V + C_2 e^{-2\mu t} \tag{7}$$

where $C_3 = C_1/\lambda_{\max}(P_\infty)$. This implies that $\lim_{t\to\infty} x = \boldsymbol{x}_\infty$. The steady differentiator gain is a high-gain vector of the following form:

$$P_\infty C' = \begin{bmatrix}\alpha_1\gamma & \alpha_2\gamma^2 & \cdots & \alpha_n\gamma^n\end{bmatrix}' \tag{8}$$

Take the Laplace transform of the first of Eqs. (4) and let $\mathscr{X}_\infty(s)$ and $Y(s)$ be the Laplace transform of $\boldsymbol{x}_\infty$ and y, respectively; then

$$\mathscr{X}_\infty(s)/Y(s) = (sI - A + P_\infty C'C)^{-1}P_\infty C' \tag{9}$$

Dividing the numerator and the denominator of the each transfer function of $\mathscr{X}_\infty(s)/Y(s)$ by γ^n, we get for γ sufficiently large

$$\begin{aligned}\left[\frac{\mathscr{X}_\infty(s)}{Y(s)}\right]_j &= \sum_{k=j}^{n}\alpha_k\gamma^{k-n}s^{n-k+j-1}\Bigg/\sum_{k=0}^{n}\alpha_k\gamma^{k-n}s^{n-k} \\ &= \left(\alpha_n s^{j-1} + \sum_{k=j}^{n-1}\frac{\alpha_k}{\gamma^{n-k}}s^{n-k+j-1}\right)\Bigg/\left(\alpha_n + \sum_{k=0}^{n-1}\frac{\alpha_k}{\gamma^{n-k}}s^{n-k}\right) \\ &\simeq s^{j-1}\end{aligned} \tag{10}$$

It means that the jth state x_j approximates the $(j-1)$th derivative of y. This ends the proof.

Discrete-Time Differentiation Observer

Because the Kalman filter is generally implemented on digital computers, this section concerns the development of the discrete-time model of the differentiation observer. The proposed discrete-time differentiator is formulated as a classical Kalman filter, written in a predictor–corrector form. The breakdown of the derivative estimates is summarized in the following theorem.

Theorem 2: Consider the discrete-time system

$$x_{k+1} = e^{A\delta}x_k + P_kC'[(1/\delta) + CP_kC']^{-1}\big(y_k - Ce^{A\delta}x_k\big)$$
$$P_{k+1} = e^{A\delta}P_ke^{A'\delta} - e^{A\delta}P_kC'[(1/\delta) + CP_kC']^{-1}CP_ke^{A'\delta} + Q(\beta,\delta) \tag{11}$$

where A and C are defined as in Theorem 1 and the matrix Q is defined as follows:

$$Q(\beta,\delta) = \mathrm{diag}\big[\beta_1/\delta, \beta_2/\delta^3, \ldots, \beta_n/\delta^{2n-1}\big] \qquad (\beta_i, i = 1,\ldots,n) \in \mathbb{R}_+$$

Then for sufficiently small sampling period δ and for any bounded signal y_k, $k \in \mathbb{Z}_+$ of class $\mathscr{C}^{(n)}$ such that 1) $[\sup_{k\geq 0}|y_k^{(i)}| \leq (1/\delta)$, $i = 1, \ldots, n]$, and 2) the polynomial

$$s^n + \sum_{i=1}^{n} \alpha_i s^{n-i}$$

is Hurwitz, the vector state of Eq. (11) approximates the successive higher derivatives of y_k for all $k \in \mathbb{Z}_+$. The vector α is defined as in theorem 1 and $(\beta_i = \alpha_i^2 - 2\alpha_{i+1}\alpha_{i-1} + 2\alpha_{i+2}\alpha_{i-2} - \cdots)$, $1 \leq i \leq n$.

Proof: When δ is too small, we can write the following approximations:

$$[(1/\delta) + CP_kC']^{-1} \simeq \delta$$

$$e^{A\delta} = I + \delta A + (\delta^2/2!)A^2 + (\delta^3/3!)A^3 + \cdots \simeq I + \delta A$$

$$Ce^{A\delta} \simeq C \tag{12}$$

Then

$$x_{k+1} = (I + \delta A)x_k + \delta P_k C'(y_k - Cx_k)$$

Consequently,

$$\lim_{\delta\to 0}[(x_{k+1} - x_k)/\delta] = \dot{x} = Ax + PC'(y - Cx)$$

Substituting approximation (12) in Eq. (11), we get

$$P_{k+1} = (I + \delta A)P_k(I + \delta A)' - \delta(I + \delta A)P_kC'CP_k(I + \delta A)'$$
$$+ Q(\beta, \delta) = P_k + \delta P_k A' + \delta A P_k + \delta^2 A P_k A'$$
$$+ \delta\left(P_kC'CP_k + \delta^2 A P_k C'CP_k\right)(I + \delta A)' + Q(\beta, \delta)$$

When all of the terms having powers of δ greater or equal to 2 are neglected, then

$$(P_{k+1} - P_k)/\delta = A\,P_k + P_kA' - P_kC'CP_k + [Q(\beta, \delta)/\delta] \tag{13}$$

If we take $\gamma = 1/\delta$, and $\beta_i = \alpha_i^2 - 2\alpha_{i+1}\alpha_{i-1} + 2\alpha_{i+2}\alpha_{i-2} - \cdots$, then

$$Q(\beta, \delta)/\delta = \mathrm{diag}\Big[\big(\alpha_i^2 - 2\alpha_{i+1}\alpha_{i-1}$$
$$+ 2\alpha_{i+2}\alpha_{i-2} - \cdots\big)\gamma^{2i}; \quad i = 1, \ldots, n\Big]$$

When the limit $\lim_{\delta\to 0}(P_{k+1} - P_k)/\delta$ is taken, the discrete equation (13) coincides with the continuous-time differential equation given in Eq. (2). Hence, as we have proved Theorem 1, we conclude that system (11) is a discrete-time differentiation observer.

Note that when time elapses the discrete-time differentiator reduces to a time-invariant filter. This is equivalent to an optimal discrete filter, which in a transfer function form is known as a Wiener filter. The discrete vector gain $P_\infty C'[(1/\delta) + CP_\infty C']^{-1}$ takes the form $[\lambda_1 \;\; \lambda_2/\delta \cdots \lambda_n/\delta^{n-1}]'$, where λ_i, $i = 1, \ldots, n$, are nonlinear functions of the parameters β_i, $i = 1, \ldots, n$. Then the model of the discrete-time differentiator is

$$x_{k+1} = e^{A\delta}x_k + \begin{bmatrix} \lambda_1 \\ \lambda_2/\delta \\ \vdots \\ \lambda_n/\delta^{n-1} \end{bmatrix} \left(y_k - Ce^{A\delta}x_k\right) \tag{14}$$

For $n = 2$, the discrete time differentiator (11) is exactly the well-known α–β filter, here, $\alpha = \lambda_1$ and $\beta = \lambda_2$,

$$x_1(k+1) = x_1(k) + \delta x_2(k) + \lambda_1[y(k) - \bar{y}(k)]$$
$$x_2(k+1) = x_2(k) + (\lambda_2/\delta)[y(k) - \bar{y}(k)] \tag{15}$$

where $\bar{y}(k) = CAx(k)$.

Application to Target Tracking

We suppose that no mathematical model is assumed to be known for the target dynamics. The objective is to estimate the velocity and the acceleration of the target in Cartesian coordinates x and y (Fig. 1). We suppose that the target flies at a constant but unknown speed along a circular trajectory of radius $r = 3000$ m. The center of the target trajectory is located at $x = y = 5000$ m and $z = 1000$ m. This flight maneuver generates sinusoidal functions of time along x and y coordinates. This gives $x = r\cos\omega t$, $\dot{x} = -r\omega\sin\omega t$, and $\ddot{x} = -r\omega^2\cos\omega t$ and $y = r\sin\omega t$, $\dot{y} = r\omega\cos\omega t$, and $\ddot{y} = -r\omega^2\sin\omega t$. We suppose that

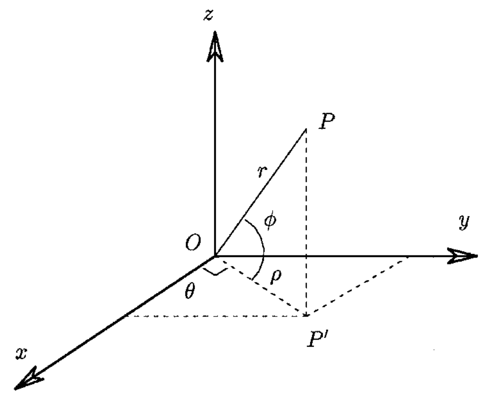

Fig. 1 Target tracking.

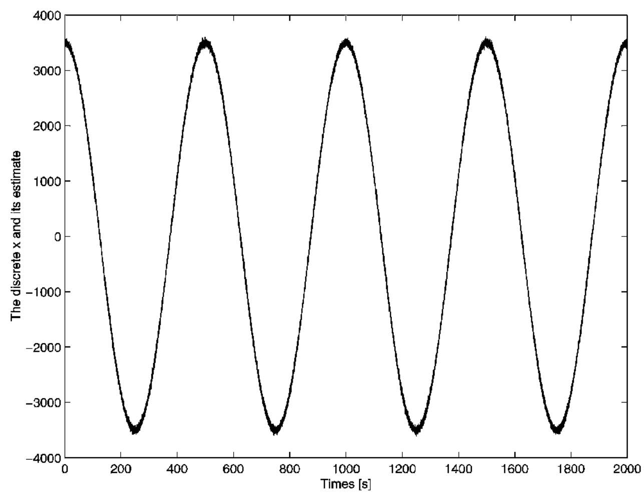

Fig. 2 Discrete x and its estimate $\hat{x}$.

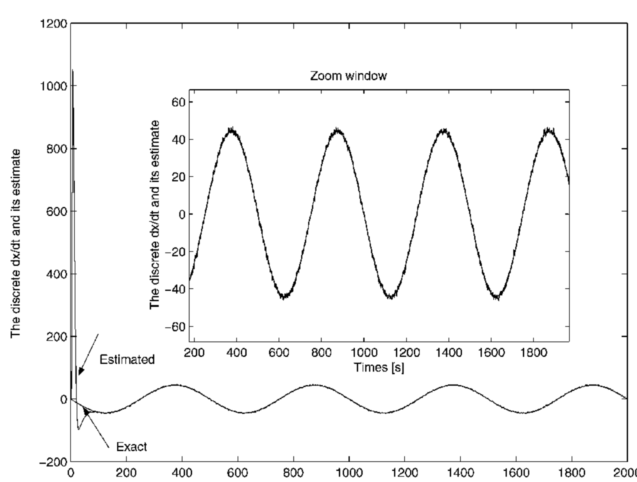

Fig. 3 Discrete $\dot{x}$ and its estimate $\hat{\dot{x}}$.

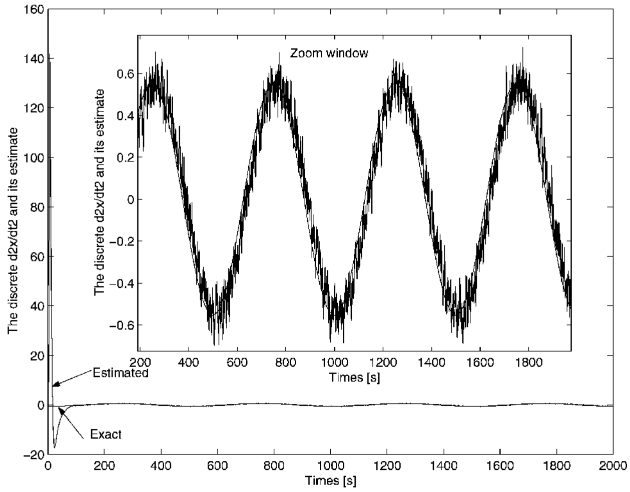

Fig. 4 Discrete $\ddot{x}$ and its estimate $\hat{\ddot{x}}$.

x and y are totally determined by measuring r, θ, and ϕ in a discrete manner. To compute the estimates of the velocities and the accelerations, $v_x = \dot{x}$ and $a_x = \ddot{x}$, $v_y = \dot{y}$, and $a_y = \ddot{y}$, we write two dynamic systems of order three in which the measurement x and y appear as inputs of the differentiation models. The parameter γ is selected so that $\gamma > r\omega^2$. System (11) is employed for $n = 3$ to estimate the first discrete derivatives of x and y with an appropriate sampling period $\delta = 10^{-3}$ s and $\alpha_i = C_3^i$, $i = 1, 2, 3$ (Figs. 2–4). The intensity of noise is set equal to 10% of the amplitude of the measured signal. In the discrete-time case, the tradeoff between the fast convergence of the estimates and filtering is a difficult task. The best coefficients of the discrete-time differentiator and the sampling period must be chosen according to the noise level. Notice that in the continuous-time case, the differentiator works perfectly and the differentiation error can be reduced more by increasing the value of γ. One can easily prove that the differentiation error is proportional to $1/\gamma$ (Ref. 3).

Conclusions

In this Note a continuous-time arbitrary-order differentiator is proposed to estimate the time derivatives of any bounded differentiable signal. The differentiation design strategy is formulated as a Kalman observer, where the model of the signal does not appear in the differential equations of the observer. The discrete-time version of the differentiator is also discussed to deal with sampled signals. The proposed discrete-time differentiation observer generalizes all Kalman suboptimal trackers. These kinds of differentiators can be applied in many fields of research, such as multiple target tracking, stabilization problems, identification, and observer design, in which no dynamic mathematical model is explicitly assumed.

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